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SOLUTIONS BOOK

CLASS 9

Mathematics · Ganita Manjari

CHAPTER 12

Quadrilaterals

Complete Exercise Solutions

Exercise Sets 12.1 – 12.4 · End-of-Chapter

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IT

CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

In-Text Questions

Think and Reflect · Activity · (Why?) and (How?) prompts

Directions: These are the questions that appear inside the running text of the chapter (the *Think and Reflect* boxes, the activity and the "Is this converse true?", "(Why?)" and "(How?)" prompts). Each is answered below in the order in which it appears; the page number of the chapter is given with every question.

IN-TEXT 1 · P. 51

Think and Reflect. Can we use any given quadrilateral to tile the plane? If not, which quadrilaterals can be used and which cannot?

ANSWER

Y **Can any quadrilateral tile the plane?** YES

Yes. **Every** quadrilateral, convex or non-convex, tiles the plane, so no quadrilateral has to be excluded.

Parallelograms: a grid of two sets of parallel sticks (Fig. 12.1) gives copies of any fixed parallelogram with no gaps or overlaps.

Any 4-gon SOME: its four angles add up to 360°, so four copies can be fitted around a point using each angle once. Rotating a copy by 180° about the midpoint of an edge gives the neighbour across that edge, and repeating this at every edge covers the plane (Method 1 of Section 12.4; Exercise Set 12.4 Q2 shows it for a non-convex 4-gon).

Therefore, every quadrilateral tiles the plane; there is no quadrilateral that cannot be used.

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IN-TEXT 2 · P. 52

Think and Reflect. Informally, a quadrilateral is a figure with four straight sides, as in the first figure ABCD in Fig. 12.2 below. But consider the other six figures in Fig. 12.2: the five plane figures NOPE, SILY, DART, CUTS, OPENS and the non-planar BENT. Should we call all these figures quadrilaterals? If you answer 'no' for any of them, how will you define a quadrilateral so that such a figure is excluded? As you can see, some care is needed to precisely define what we think of as a quadrilateral.

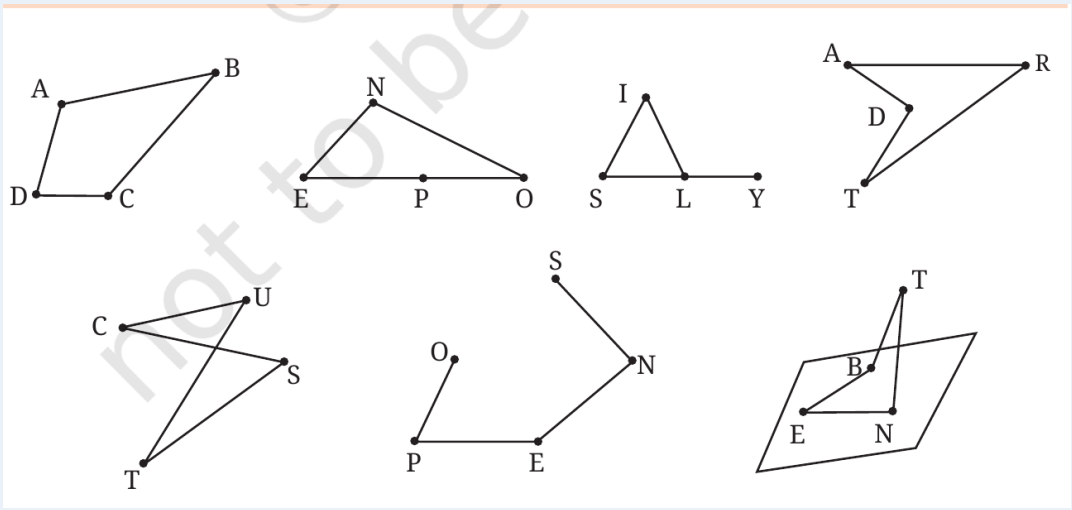

Fig. 12.2

ANSWER

1 NOPE — NO

N, E, P, O are such that E, P, O are collinear. Sides EP and PO lie along one line, so the figure is just the triangle NEO with an extra point P on a side.

2 SILY — NO

S, L, Y are collinear. The sides through them lie along the same line and overlap, so we do not get a genuine four-sided figure.

3 DART — YES

DART has four vertices, no three collinear, lies in a plane and does not cross itself. It only has a dent at D, so it is a **non-convex** quadrilateral.

4 CUTS — NO

Two sides (UT and CS) cross each other, so it is self-intersecting. Ordinarily this is not called a quadrilateral (if we wish to include it we call it a self-intersecting quadrilateral).

5 OPENS — NO

It is an open chain of segments: the figure does not close up, and it also has more than four vertices.

6 BENT — NO

Its four vertices are not in one plane (non-planar).

not to be
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Definition 1. Four distinct points A, B, C, D in a plane; the points on the 4 segments AB, BC, CD, DA form a quadrilateral if every such point other than A, B, C, D lies on exactly one of these four segments.

Therefore, only ABCD and DART are quadrilaterals; NOPE, SILY, CUTS, OPENS and BENT are excluded by Definition 1.

IN-TEXT 3 · P. 53

Think and Reflect. Can we similarly define a quadrilateral ABCD?

Take any 4 distinct non-collinear points A, B, C and D. Will segments AB, BC, CD and DA always form a quadrilateral as we visualise it? … Can you see how to rule out figures like OPENS?

ANSWER

No: four points alone are not enough, because the four segments can produce the unwanted figures NOPE, SILY, BENT, CUTS and OPENS. Each needs a separate condition:

a **NOPE and SILY**

No three of the vertices may be collinear.

b **BENT**

All four vertices must lie in one plane.

c **CUTS**

Every point other than the vertices must lie on exactly one side, so that the sides do not cross.

d **OPENS**

Use exactly four vertices and all four segments AB, BC, CD and DA, so that the figure closes up: every vertex is the endpoint of exactly two sides. An open chain such as OPENS has end points that belong to only one segment, and it has five vertices.

Putting (a), (b), (c) and (d) together gives Definition 1 of the chapter.

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IN-TEXT 4 · P. 57

(a) If a quadrilateral is a parallelogram, then its opposite sides are equal. **Converse:** If the opposite sides of a quadrilateral are equal, then it is a parallelogram. Is this converse statement true?

(b) If a quadrilateral is a parallelogram, then the opposite angles are equal. **Converse:** If the opposite angles of a quadrilateral are equal, then it is a parallelogram. Is this converse statement true?

(c) If a quadrilateral is a parallelogram, then its diagonals bisect each other. **Converse:** If the diagonals of a quadrilateral bisect each other, then it is a parallelogram. Is this converse statement true?

ANSWER

a **Converse of (a)** TRUE

Let ABCD have AB = DC and AD = BC. Then $\Delta ACD \cong \Delta CAB$ by SSS (AC is common). So $\angle CAB = \angle ACD$ and $\angle CAD = \angle ACB$. Using the transversal AC, these equal alternate angles give AB ∥ DC and AD ∥ BC.

b **Converse of (b)** TRUE

Let $\angle A = \angle C$ and $\angle B = \angle D$. Since the angles add up to 360°, $2(\angle A + \angle B) = 360°$, so $\angle A + \angle B = 180°$ and likewise $\angle B + \angle C = 180°$. Co-interior angles are supplementary, so BC ∥ AD and AB ∥ DC.

c **Converse of (c)** TRUE

Let the diagonals AC and BD meet at E with EA = EC and EB = ED. $\Delta AED \cong \Delta CEB$ by SAS (vertically opposite angles at E), so $\angle DAE = \angle ECB$ and AD ∥ BC. Similarly $\Delta EAB \cong \Delta ECD$ gives AB ∥ DC.

Therefore, all three converse statements are true.

IN-TEXT 5 · P. 59

Think and Reflect. The angles of a quadrilateral add up to 360°. Therefore, if the opposite angles are equal, what can we say about adjacent angles? Is the converse of your answer true? Conclude that Theorem 3 can also be stated as follows. "If each pair of adjacent angles in a quadrilateral ABCD … then ABCD is a parallelogram." Fill in the blank.

ANSWER

If $\angle A = \angle C$ and $\angle B = \angle D$, then $2(\angle A + \angle B) = 360°$, so $\angle A + \angle B = 180°$. In the same way every pair of adjacent angles adds up to 180°. So **adjacent angles are supplementary**.

Converse: if every two adjacent angles are supplementary, then the opposite angles are equal. Indeed $\angle A + \angle B = 180° = \angle B + \angle C$ gives $\angle A = \angle C$, and $\angle B + \angle C = 180° = \angle C + \angle D$ gives $\angle B = \angle D$. So the converse is true.

Blank: "… are supplementary (add up to 180°), then ABCD is a parallelogram."

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IN-TEXT 6 · P. 61

Activity. On a piece of paper, draw and then cut out two copies of the same triangle. Keep one copy aside. On the other, mark the midpoint of each side. Draw the triangle made by the three midpoints and cut the paper along each of these lines. Now you have four smaller triangles. Compare them to each other and to the intact copy of the original triangle. What do you notice?

ANSWER

The four small triangles fit exactly on top of one another, so they are **congruent to each other**.

Each small triangle is a smaller copy of the original: its sides are exactly **half** as long as the sides of the original triangle, its angles are the same as those of the original, and its area is **one-fourth** of the original.

This pattern is true for every triangle and is proved in Exercise Set 12.3, Q1 using the Midpoint Theorem.

IN-TEXT 7 · P. 62

Think and Reflect. Try to prove directly that the four smaller triangles are congruent. You will see that no congruence test applies, because many quantities are unknown.

ANSWER

In ΔABC with midpoints P, Q, R of AB, AC, BC, all we know directly are the equalities AP = PB, AQ = QC and BR = RC. The sides PQ, QR, RP and the angles of the small triangles are not given, so SSS, SAS, ASA and AAS cannot be used.

The Midpoint Theorem supplies what is missing: PQ = BC/2, QR = AB/2 and PR = AC/2. Then ΔAPQ, ΔPBR, ΔQRC and ΔRQP all have sides AB/2, BC/2 and AC/2, so they are congruent by SSS.

Therefore, the Midpoint Theorem is the missing step; see Exercise Set 12.3, Q1 for the full argument.

IN-TEXT 8 · P. 64

In the proof of the converse of the Midpoint Theorem: “BCRP is a parallelogram, so CR = BP. BP = PA and ∠AQP = ∠CQR (vertically opposite angles). Combining with CR ∥ BA, ΔAPQ ≅ ΔCRQ. (Why?)”

ANSWER

From BCRP being a parallelogram, CR = BP, and BP = PA because P is the midpoint of AB. Hence **CR = AP**.

Since CR ∥ BA, the transversal PR gives ∠APQ = ∠CRQ (alternate angles). Also ∠AQP = ∠CQR (vertically opposite).

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In ΔAPQ and ΔCRQ: ∠APQ = ∠CRQ, ∠AQP = ∠CQR and AP = CR, so ΔAPQ ≅ ΔCRQ by the AAS test.

IN-TEXT 9 · P. 65

In Grade 10, we will generalise Theorem 7 by dropping the condition that P is the midpoint of AB and assuming only that PQ ∥ BC. Can you guess the conclusion in this case? (Hint: the name of the more general theorem includes the word proportionality.)

ANSWER

If P is a point on AB and Q a point on AC with PQ ∥ BC, then the line PQ divides the two sides in the same ratio:

$$AP / PB = AQ / QC$$

This is the **Basic Proportionality Theorem**. For P the midpoint of AB it gives AQ = QC, which is Theorem 7. Exercise 11(iv) of the end-of-chapter exercises proves it when AP/PB is rational.

Therefore, the conclusion is AP : PB = AQ : QC.

IN-TEXT 10 · P. 66

In the proof of the Centroid Theorem: “So ΔMPQ ≅ ΔMYX by ASA (How?). Therefore MQ = MX = XB and MP = MY = YC.”

ANSWER

Here P, Q are the midpoints of AB, AC, and X, Y the midpoints of BM, CM. The Midpoint Theorem gives PQ ∥ BC and XY ∥ BC (so PQ ∥ XY), and PQ = BC/2 = XY.

Since PQ ∥ XY, the transversal PY gives ∠MPQ = ∠MYX and the transversal QX gives ∠MQP = ∠MXY (alternate angles).

In ΔMPQ and ΔMYX: ∠MPQ = ∠MYX, PQ = YX, ∠MQP = ∠MXY, so ΔMPQ ≅ ΔMYX by ASA. Hence MQ = MX, MP = MY; with X, Y midpoints of BM, CM we get MQ = MX = XB and MP = MY = YC.

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IN-TEXT 11 · P. 66

Think and Reflect. Give another proof of the Centroid Theorem using a clever construction that we will describe only partly and ask you to complete. Once again let M be the intersection point of the two medians CP and BQ. We want to show that the intersection point of line AM with side BC is the midpoint of BC. For this, extend AM up to a carefully chosen point S such that BSCM becomes a parallelogram. (Hint: To arrange MC ∥ BS, first note that PM and MC are along the same line. Now where should M be on segment AS? Does your placement of S also prove MB ∥ CS?)

ANSWER

Choice of S. Take S on ray AM with **MS = AM**, i.e. M is the midpoint of AS.

MC ∥ BS. In ΔASB, P is the midpoint of AB (it is on median CP) and M is the midpoint of AS, so by the Midpoint Theorem PM ∥ BS. But PM lies along line CM, hence MC ∥ BS.

MB ∥ CS. In ΔASC, Q is the midpoint of AC and M the midpoint of AS, so QM ∥ CS. QM lies along line BM, hence MB ∥ CS.

So BSCM has both pairs of opposite sides parallel: it is a parallelogram. Its diagonals BC and MS bisect each other, so they meet at a point X which is the midpoint of BC and also of MS. Thus AM extended meets BC at its midpoint X (the third median passes through M).

Moreover MX = MS/2 = AM/2, so AM : MX = 2 : 1.

Therefore, the three medians are concurrent at M and each is divided 2 : 1 by M (the longer part towards the vertex).

IN-TEXT 12 · P. 70-71

Think and Reflect. Suppose we have a tiling of the plane. Consider any vertex. As we go around this vertex and consider the angles made by consecutive lines, the total of these angles must be 360°. This suggests an idea. What if we take 4 copies of SOME and fit them together around a common point so that each angle is used once as we go around?

There are multiple ways of doing this, as shown in Fig. 12.27. Can you use any of these ways to continue fitting further copies of SOME to tile the plane? Try it with the 15 copies you made!

Method 1. ... Verify that this will result in a tiling! ... (1) Can you precisely describe a procedure to draw the pattern so that someone can draw the tiling on their own, based only on your description? (2) Can you justify why your procedure works? ... Do you see which of the three possibilities shown in Fig. 12.27 occurs in the tiling?

ANSWER

Four copies around a point. The angles 1, 2, 3, 4 of SOME add up to 360°, so four copies can meet at a point with each angle used exactly once. They fill the whole 360°, with no gap and no overlap.

1 Procedure (Method 1)

(a) Draw SOME. (b) Across each edge, draw the copy obtained by rotating SOME through 180° about the midpoint of that edge (it fits exactly along the edge). (c) Repeat (b) for every edge of every copy that is added. Two steps give Fig. 12.30 A and B, and continuing fills the plane.

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2 Why it works

Take the vertices in order $V_0V_1V_2V_3$ and let H be the half-turn about the midpoint of V_0V_1. Write $d_1 = V_2 - V_0$ and $d_2 = V_3 - V_1$ (the diagonals as vectors). The copy across the edge V_1V_2 is the half-turn image about its midpoint, which equals H(SOME) shifted by d_1; across V_2V_3 it equals H(SOME) shifted by $d_1 + d_2$; across V_3V_0 it equals H(SOME) shifted by d_2. Hence the whole pattern consists of the copies SOME and H(SOME), each shifted by all combinations $i \cdot d_1 + j \cdot d_2$ (i, j integers); every neighbour of a copy is again in the pattern, so the new copies fit perfectly and each can be obtained from any one of its neighbours by a half-turn.

At each vertex of the pattern, two copies of the first kind and two of the second kind meet, and between them they carry the angles 1, 2, 3, 4 once each, which add up to 360°. The pattern has no gaps (the pieces fit around every vertex and along every edge) and no overlaps (the area of one cell of the grid $i \cdot d_1 + j \cdot d_2$ is $|d_1 \times d_2| = 2 \times$ area of SOME, exactly the area of the two copies in it). See also Exercise 23.

3 Which arrangement of Fig. 12.27?

Going once around any vertex of the tiling, the angles occur in the cyclic order 1, 2, 3, 4: from a copy we pass across an edge to its half-turn image, which brings the next angle to the vertex. So the tiling uses the arrangement of Fig. 12.27 in which the four angles meet in the order 1, 2, 3, 4 and the neighbouring copies are related by half-turns about edge midpoints.

Therefore, copies of any 4-gon fitted around a point in this way continue to tile the whole plane.

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12.1

CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

Exercise Set 12.1

Solutions · 5 questions

QUESTION 1

Let ABCD be a quadrilateral.

(i) List all sides of ABCD adjacent to side AB. List all sides opposite to AB.

(ii) List all angles of ABCD adjacent to ∠A. List all angles opposite to ∠A.

(iii) Define a pair of opposite sides and a pair of opposite angles without using the names of the vertices.

ANSWER

i Sides

Sides adjacent to AB (they share an endpoint with AB): **BC and DA**. Side opposite to AB (no common endpoint): **CD**.

ii Angles

Angles adjacent to ∠A (at the adjacent vertices B and D): **∠B and ∠D**. Angle opposite to ∠A: **∠C**.

iii Definitions

Opposite sides: two sides of a quadrilateral that do not have a common endpoint (they do not meet at a vertex).

Opposite angles: two internal angles whose vertices are not adjacent, i.e. they do not lie at the two endpoints of the same side (they lie at the two ends of a diagonal).

Therefore, the answers are: (i) BC, DA; CD. (ii) ∠B, ∠D; ∠C. (iii) as defined above.

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QUESTION 2 *

You have used internal angles of quadrilaterals, but they too require an exact definition, just like how we gave one for a quadrilateral. Precisely define the internal angle of a quadrilateral at a given vertex. Your answer should work for a non-convex quadrilateral too. (Hint: use the opposite vertex as well.)

ANSWER

Let V be a vertex of quadrilateral VUXW with adjacent vertices U and W, and opposite vertex X. The two rays VU and VW divide the plane into two angles: the ordinary angle ∠UVW (less than 180°) and the reflex angle 360° − ∠UVW.

> **Definition.** The internal angle at V is the ordinary angle ∠UVW (< 180°), unless V lies inside the triangle formed by the other three vertices U, W, X; in that case the internal angle at V is the reflex angle 360° − ∠UVW (> 180°).

Check. For a convex quadrilateral no vertex lies inside the triangle of the other three, so every internal angle is the ordinary angle. For the dart DART, the dent D lies inside ΔART, so the internal angle at D is 360° − ∠ADT, which is more than 180°; the other three vertices lie outside the triangle of the remaining three, so their angles are ordinary angles.

Therefore, the internal angle at V is the angle between the two sides at V that faces the inside of the quadrilateral, as defined above.

QUESTION 3

In a quadrilateral ABCD, suppose AB ∥ DC. Can ABCD be non-convex? What if we instead assume AB = CD? What if we instead assume ∠A = ∠C?

ANSWER

(a) If AB ∥ DC, can ABCD be non-convex? NO

Take A, B on a line m and D, C on a parallel line l, with ABCD not self-intersecting. Then AB and DC point the same way (if they pointed opposite ways, sides AD and BC would cross). Take coordinates with l: y = 0, m: y = 1, A = (a, 1), B = (b, 1), D = (d, 0), C = (c, 0), a < b and d < c. On the line at height t the diagonals AC and BD are at x = c + (a − c)t and x = d + (b − d)t. Their difference is c − d > 0 at t = 0 and a − b < 0 at t = 1, so it vanishes for some t between 0 and 1: the diagonals intersect.

Diagonals that intersect mean the quadrilateral is convex. So **ABCD cannot be non-convex** (this also shows why the diagonals of such a quadrilateral intersect, used in Theorem 5).

(b) If AB = CD, can ABCD be non-convex? YES

Counterexample: a non-convex quadrilateral with AB = CD.

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Take A = (0, 0), B = (10, 0), C = (6, 2), D = (0, 10). Then AB = 10 and CD = $\sqrt{(6^2 + 8^2)} = 10$, so AB = CD. C lies inside ΔABD (6/10 + 2/10 < 1), so no sides cross and ABCD has a dent at C: it is non-convex.

c **If ∠A = ∠C, can ABCD be non-convex?** YES

Counterexample: the dart DART (symmetric about the diagonal DR).

In DART, A and T are opposite vertices and are mirror images in the line DR, so ∠A = ∠T, while the quadrilateral is non-convex (dent at D).

Therefore, AB ∥ DC forces convexity, but AB = CD or ∠A = ∠C alone does not.

QUESTION 4

Consider three non-collinear points A, B, C and draw the lines AB, BC, CA. For every possible location of point D in the plane outside these lines, decide if ABCD is self-intersecting, non-convex, or convex. (Hint: the three lines divide the plane into 7 regions.)

ANSWER

The 7 regions are: the inside of ΔABC (1 region), the three regions across each side (touching a side), and the three vertical-angle regions at the vertices A, B, C. In ABCD the sides are AB, BC, CD, DA, with AC and BD the diagonals.

1 **D inside ΔABC** NON-CONVEX

D lies inside the triangle formed by the other three vertices; no sides cross. The internal angle at D is more than 180°.

2 **D in the region across AC (opposite to B, inside ∠ABC)** CONVEX

D and B are on opposite sides of AC and A, C on opposite sides of BD, so the diagonals AC and BD cross: ABCD is convex.

3 **D in the region across AB (opposite to C)** SELF-INTERSECTING

Then segment CD crosses side AB, i.e. the opposite sides AB and CD cross.

4 **D in the region across BC (opposite to A)** SELF-INTERSECTING

Then segment AD crosses side BC, i.e. the opposite sides DA and BC cross.

5 **D in a vertical-angle region at A, at B or at C** NON-CONVEX

If D is in the vertical-angle region at A, then A lies inside ΔBCD (A is a positive-weight average of B, C, D), and no sides cross, so ABCD has a dent at A. The same argument gives a dent at B or at C for the regions at B and C.

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Therefore, ABCD is convex for the one region across AC; self-intersecting for the two regions across AB and BC; and non-convex for the other four regions (inside, and the three vertical-angle regions).

QUESTION 5

Can a quadrilateral be both self-intersecting and non-planar?

ANSWER

A self-intersecting quadrilateral has two sides that cross at a point which is not a vertex. Adjacent sides meet only at their common vertex, so the crossing sides are opposite sides, say AB and CD, meeting at a point E.

Two distinct lines that meet at a point lie in one plane. So the lines AB and CD lie in a common plane, and this plane contains all four vertices A, B, C, D. Hence the quadrilateral is planar.

Therefore, no: a self-intersecting quadrilateral is always planar, so it cannot be non-planar.

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12.2

CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

Exercise Set 12.2

Solutions · 4 questions

QUESTION 1

True or false?

(i) A parallelogram with a right angle is a rectangle.

(ii) A rhombus with perpendicular diagonals is a square.

(iii) If the diagonals of a parallelogram are equal, then it is a rectangle.

ANSWER

i A parallelogram with a right angle is a rectangle — TRUE

In a parallelogram adjacent angles add up to 180°. If one angle is 90°, then its two adjacent angles are 90°, and the fourth angle (opposite, hence equal) is also 90°. A parallelogram with all angles 90° is a rectangle.

ii A rhombus with perpendicular diagonals is a square — FALSE

The diagonals of every rhombus are perpendicular, so this condition says nothing new; it would make every rhombus a square.

Counterexample: a rhombus of side 5 cm with diagonals 6 cm and 8 cm. The diagonals are perpendicular, but the angles are not 90°, so it is not a square.

iii If the diagonals of a parallelogram are equal, then it is a rectangle — TRUE

Let ABCD be a parallelogram with AC = BD. In ΔABC and ΔDCB: AB = DC (opposite sides), BC = CB (common) and AC = DB. So ΔABC ≅ ΔDCB by SSS and ∠ABC = ∠DCB. As AB ∥ DC, ∠ABC + ∠DCB = 180°, so each is 90°. A parallelogram with a right angle is a rectangle.

Therefore, (i) true, (ii) false, (iii) true.

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QUESTION 2

The diagonal AC of a parallelogram ABCD bisects ∠A. Show that it also bisects ∠C and that ABCD is a rhombus.

ANSWER

Given ∠DAC = ∠CAB. Since AB ∥ DC with transversal AC, ∠CAB = ∠ACD (alternate angles). Since AD ∥ BC with transversal AC, ∠DAC = ∠ACB (alternate angles).

$$\angle ACD = \angle CAB = \angle DAC = \angle ACB$$

So ∠ACD = ∠ACB, i.e. **AC bisects ∠C**.

In ΔADC, ∠DAC = ∠ACD, so the sides opposite them are equal: DC = AD. In a parallelogram AB = DC and AD = BC, hence AB = BC = CD = DA.

Therefore, AC also bisects ∠C and all four sides are equal, so ABCD is a rhombus.

QUESTION 3

The following questions examine converses of true properties. Answer them with Yes or No. If your answer is No, what extra condition can you add so that the answer becomes Yes?

(i) If the diagonals of a quadrilateral ABCD bisect its angles, must ABCD be a rhombus?

(ii) If the diagonals of a quadrilateral ABCD bisect each other at right angles, must ABCD be a rhombus?

(iii) If the diagonals of a quadrilateral ABCD are of equal length, must ABCD be a rectangle? (In the ancient Indian study of quadrilaterals, equality of diagonals was considered significant. Quadrilaterals were first classified according to whether their diagonals were equal or not, before considering equality of sides.)

ANSWER

i Diagonals bisect the angles — YES

AC bisects ∠A and ∠C, so in ΔABC and ΔADC: ∠BAC = ∠DAC, ∠BCA = ∠DCA and AC is common. Hence ΔABC ≅ ΔADC (ASA) and AB = AD, CB = CD. Similarly BD bisects ∠B and ∠D, giving ΔBAD ≅ ΔBCD, so AB = CB and AD = CD. Therefore AB = BC = CD = DA: ABCD is a rhombus.

ii Diagonals bisect each other at right angles — YES

Diagonals that bisect each other make ABCD a parallelogram (Theorem 4). Let them meet at E. In ΔAEB and ΔAED: AE = AE, EB = ED and ∠AEB = ∠AED = 90°, so ΔAEB ≅ ΔAED (SAS) and AB = AD. A parallelogram with two adjacent sides equal is a rhombus.

iii Diagonals of equal length — NO

Counterexample: an isosceles trapezium with A(0, 0), B(10, 0), C(7, 4), D(3, 4). Its diagonals AC = BD = √65, but it is not a rectangle (opposite sides AB = 10 and CD = 4 are unequal).

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Extra condition: if the equal diagonals also **bisect each other** (or if ABCD is a parallelogram), then ABCD is a parallelogram with equal diagonals, hence a rectangle (Q1(iii)).

Therefore, (i) Yes, (ii) Yes, (iii) No; adding "the diagonals bisect each other" makes (iii) true.

QUESTION 4

Let ABCD be a parallelogram with AB ≠ BC. Show that the pairwise intersection points of the four angle bisectors form the vertices of a rectangle (see Fig. 12.11). Why did we assume AB ≠ BC?

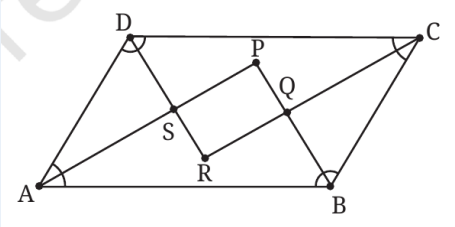

Fig. 12.11

ANSWER

Let the bisectors of the angles at the adjacent vertices A and D meet at S, those at D and C at P, those at C and B at Q, and those at B and A at R.

In a parallelogram ∠A + ∠D = 180°, so ½∠A + ½∠D = 90°. In ΔASD, ∠SAD + ∠SDA = 90°, so ∠ASD = 90°. In the same way ∠DPC = ∠CQB = ∠BRA = 90°.

The sides of quadrilateral PQRS lie along the bisectors. At S the sides SP (on the bisector of ∠D) and SR (on the bisector of ∠A) are perpendicular, so ∠PSR = 90°. Similarly the angles at P, Q and R are 90°.

A quadrilateral with all four angles equal to 90° is a rectangle.

Why AB ≠ BC? If AB = BC, ABCD is a rhombus and the bisectors of ∠A and ∠C both coincide with the diagonal AC, while those of ∠B and ∠D coincide with BD. Then all four intersection points coincide at the centre, and there is no rectangle (the figure shrinks to a point). When AB ≠ BC the bisectors of ∠A and ∠C are distinct parallel lines, so the four points are distinct.

Therefore, P, Q, R, S are the vertices of a rectangle when AB ≠ BC.

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12.3

CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

Exercise Set 12.3

Solutions · 5 questions

QUESTION 1

(i) If P, Q, R are the midpoints of sides AB, AC, BC respectively of ΔABC, show that ΔPQR is congruent to ΔQPA and to two other triangles which you should identify.

(ii) Suppose someone erases ΔABC, leaving only ΔPQR on the paper. Can you reconstruct ΔABC from ΔPQR?

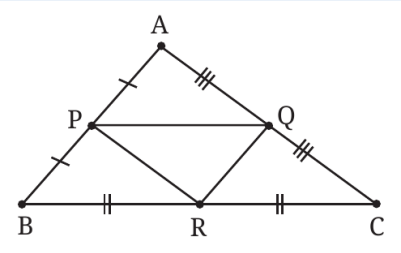

Fig. 12.22

ANSWER

i Congruent triangles

By the Midpoint Theorem: PQ = BC/2, QR = AB/2, PR = AC/2. Also AP = PB = AB/2, AQ = QC = AC/2 and BR = RC = BC/2.

ΔPQR: PQ = BC/2, QR = AB/2, RP = AC/2
ΔQPA: QP = BC/2, PA = AB/2, AQ = AC/2
ΔPBR: PB = AB/2, BR = BC/2, PR = AC/2
ΔQRC: QR = AB/2, RC = BC/2, QC = AC/2

All four triangles have sides AB/2, BC/2 and AC/2, so by SSS **ΔPQR ≅ ΔQPA ≅ ΔPBR ≅ ΔQRC**. The two other triangles are therefore **ΔPBR and ΔQRC**.

ii Reconstruct ΔABC

YES

Through each vertex of ΔPQR draw a line parallel to the opposite side: through P parallel to QR, through Q parallel to PR, through R parallel to PQ. Since QR ∥ AB, PR ∥ AC, PQ ∥ BC, these three lines are the lines AB, AC, BC. They meet pairwise at A, B, C, giving ΔABC.

Therefore, ΔPQR ≅ ΔQPA ≅ ΔPBR ≅ ΔQRC (all by SSS), and ΔABC can be reconstructed from ΔPQR.

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QUESTION 2

In ΔABC, let M and N be midpoints of AB and AC respectively. Let D be any point on BC. Show that MN bisects AD.

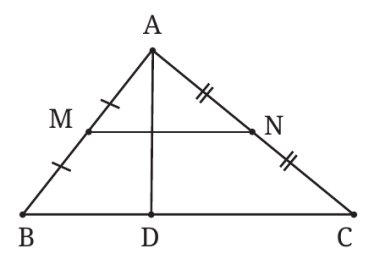

Fig. 12.23

ANSWER

Let MN meet AD at X. By the Midpoint Theorem MN ∥ BC, so MX ∥ BD.

In ΔABD, M is the midpoint of AB and MX ∥ BD. By the Converse of the Midpoint Theorem (Theorem 7), the line through M parallel to BD bisects the third side AD, i.e. X is the midpoint of AD.

Therefore, MN bisects AD.

QUESTION 3

In a quadrilateral ABCD, suppose AB ∥ DC and AB ≠ CD. Suppose G and H are the midpoints of AC and BD respectively. Prove that GH ∥ AB. (Why did we assume AB ≠ CD?)

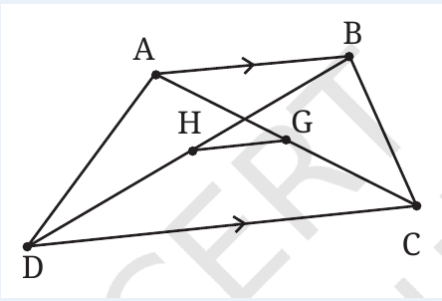

Fig. 12.24

ANSWER

Let M be the midpoint of AD.

In ΔADC, M and G are the midpoints of AD and AC, so MG ∥ DC (Midpoint Theorem) and MG = DC/2. In ΔABD, M and H are the midpoints of AD and BD, so MH ∥ AB and MH = AB/2.

Since AB ∥ DC, both MG and MH are parallel to AB and pass through M. There is only one line through M parallel to AB, so G, M, H lie on this line. Hence GH ∥ AB.

Why AB ≠ CD? Since AB and DC point the same way, MG and MH are on the same side of M with lengths DC/2 and AB/2, so GH = |AB − CD|/2. If AB = CD, then ABCD is a parallelogram, G = H (the diagonals bisect each other) and there is no segment GH.

Therefore, GH ∥ AB (and GH = |AB − CD|/2 in length).

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QUESTION 4

Suppose the midpoints of sides AB, BC, CD and DA of a quadrilateral ABCD are P, Q, R and S respectively.

(i) Show that PR and QS bisect each other.

(ii) Show that if AC = BD, then PR and QS are perpendicular. Is the converse true?

ANSWER

i PR and QS bisect each other

By Varignon's theorem (Theorem 9) PQRS is a parallelogram. PR and QS are its diagonals, and the diagonals of a parallelogram bisect each other.

ii If AC = BD then PR ⊥ QS

By the Midpoint Theorem in ΔABC, PQ = AC/2 and in ΔBCD, QR = BD/2. If AC = BD then PQ = QR, so the parallelogram PQRS has two adjacent sides equal: it is a rhombus. The diagonals of a rhombus are perpendicular, so PR ⊥ QS.

iii Converse

TRUE

If PR ⊥ QS, the parallelogram PQRS has perpendicular diagonals, so it is a rhombus (Exercise Set 12.2, Q3(ii)). Hence PQ = QR, i.e. AC/2 = BD/2, so AC = BD.

Therefore, PR and QS bisect each other; AC = BD ⇒ PR ⊥ QS, and the converse is also true.

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QUESTION 5

Suppose PQRS is the Varignon parallelogram of ABCD.

(i) Copy only PQRS on another paper. Show how you will recreate a congruent copy A′B′C′D′ of ABCD from PQRS. This will be relevant when we return to study tilings at the end of this chapter. (**Hint:** How will you place vertex A′? How will you place B′, C′ and D′?)

(ii) There are multiple ways to construct A′B′C′D′ in (i). Justify why the quadrilateral A′B′C′D′ you constructed is congruent to ABCD. You may need to show why S is collinear with the points A′ and D′ that you constructed, and similarly for P, Q and R. (**Hint:** Use congruence of triangles, for example ΔSDR ≅ ΔSD′R.)

(iii) Show that if PQRS is a square, then AC and BD are perpendicular and equal. Prove the converse.

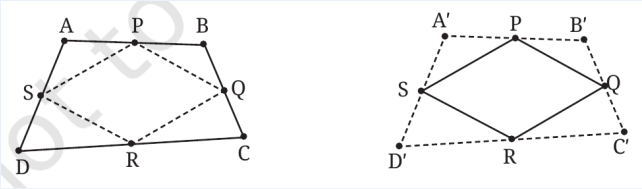

Fig. 12.25

ANSWER

i Construction

Placing A′: mark A′ on the new paper in the same position relative to PQRS as A has relative to PQRS (so that ΔA′PS ≅ ΔAPS, with A′P = AP, A′S = AS). Then:

B′: extend A′P beyond P to B′ with PB′ = PA′ (B′ is the image of A′ under a half-turn about P). **C′:** extend B′Q to C′ with QC′ = QB′. **D′:** extend C′R to D′ with RD′ = RC′. Join A′B′, B′C′, C′D′, D′A′.

ii Why A′B′C′D′ ≅ ABCD

S lies on A′D′. P, Q, R are the midpoints of A′B′, B′C′, C′D′ by construction. Let S′ be the midpoint of D′A′. By Varignon's theorem P, Q, R, S′ form a parallelogram, whose fourth vertex is fixed by P, Q, R. The parallelogram PQRS has the same three vertices P, Q, R, so S′ = S. Hence S is the midpoint of D′A′ (S, A′, D′ collinear, A′S = SD′); the same argument shows P, Q, R lie on the sides.

Congruence. The half-turns about P, Q, R carry A → B → C → D (P, Q, R are the midpoints of AB, BC, CD) and, by construction, A′ → B′ → C′ → D′. When A′ is placed like A (ΔA′PS ≅ ΔAPS), the same sequence of half-turns acts on both, so the two quadrilaterals correspond point by point: for instance ΔSDR ≅ ΔSD′R, since SR is common, SD = SD′ (S is the midpoint of both AD and A′D′ with A′S = AS) and DR = D′R (R is the midpoint of both CD and C′D′). Hence the sides and angles of A′B′C′D′ equal those of ABCD, so A′B′C′D′ ≅ ABCD.

Remark. The position of A′ is the only free choice: any A′ gives a closed quadrilateral with Varignon parallelogram PQRS, but only the placement that copies the position of A gives a copy congruent to ABCD.

iii PQRS a square ⇔ AC ⊥ BD and AC = BD

PQ ∥ AC with PQ = AC/2, and QR ∥ BD with QR = BD/2 (Midpoint Theorem). If PQRS is a square, PQ = QR so AC = BD, and PQ ⊥ QR so AC ⊥ BD.

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Converse. If AC = BD and AC ⊥ BD, then PQ = QR (so the parallelogram PQRS is a rhombus) and PQ ⊥ QR (since they are parallel to AC and BD), so ∠PQR = 90°. A rhombus with a right angle is a square.

Therefore, PQRS is a square exactly when the diagonals AC and BD of ABCD are equal and perpendicular.

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12.4

CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

Exercise Set 12.4

Solutions · 2 questions

QUESTION 1

Justify why the plane cannot be tiled with a regular pentagon. (Hint: Read the first 3 sentences of 'Think and Reflect' in the section on tiling.) (There are many ways to tile the plane using a suitable irregular pentagon. The most recent method was found in 2015.)

ANSWER

In any tiling, the angles at a point where tiles meet add up to 360° (if the point is a vertex of all the tiles meeting there), or to 180° from the tiles' vertices if the point lies on a side of another tile (the straight side supplies the other 180°).

Each interior angle of a regular pentagon is $(5 - 2) \times 180° \div 5 = \mathbf{108°}$. If k pentagons have a vertex at the point:

$$108° \times k = 360° \Rightarrow k = 10/3 \text{ (not an integer)}$$
$$108° \times k = 180° \Rightarrow k = 5/3 \text{ (not an integer)}$$

Neither is possible, so regular pentagons cannot fit together around a point without gap or overlap.

Therefore, the plane cannot be tiled with a regular pentagon.

QUESTION 2

Draw a non-convex 4-gon DART. Show how we can tile the plane with copies of DART. Both methods that we discussed earlier will work. Which do you prefer?

ANSWER

Let DART have vertices D (the dent), A, R, T in order, and let H be the half-turn about the midpoint of DA. Write $d_1 = R - D$ and $d_2 = T - A$ (the diagonals DR and AT as vectors).

1 Method 1 (half-turns)

Draw DART. Draw its copy rotated through 180° about the midpoint of one edge, so the two fit along that edge. Repeat across every edge of every new copy. The copies of DART (green) and its half-turn images (orange) form the tiling shown. The four angles of DART add up to 360° (even with the reflex angle at D), so the angles fit around each vertex exactly once.

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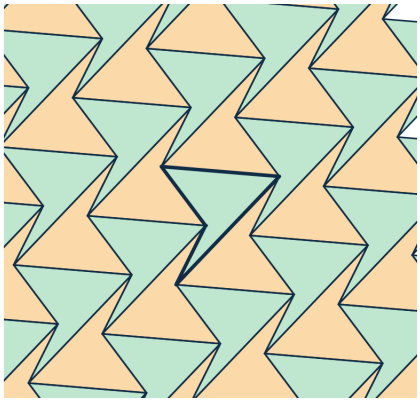

Tiling with DART: the base copy (bold), its translates (green) and half-turn images (orange)

2 **Method 2 (Varignon grid)**

Draw the Varignon parallelogram of DART (joining the midpoints of its sides). Draw the grid of parallelograms it generates. Place copies of DART in every second cell, in the same position as the original in its own parallelogram. They meet at vertices, and the gaps left are copies of DART turned through 180°, which are filled in. This gives the same tiling as Method 1.

3 **Which is preferred?**

Method 1 is simpler: it needs only a half-turn about the midpoint of an edge, with no extra construction. Method 2 first needs the midpoints, the Varignon parallelogram and its grid. (Both give identical tilings.)

Therefore, copies of the non-convex DART tile the plane, and Method 1 is the easier procedure.

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CLASS 9 MATHS · GANITA MANJARI · CHAPTER 12

End-of-Chapter Exercises

Solutions · 24 questions

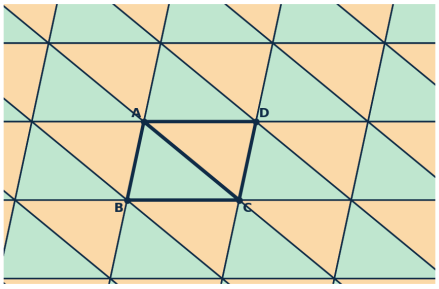
QUESTION 1

Using a fact about parallelograms, show how to tile the plane using any given triangle. (Hint: Can you use the parallelogram tiling in the introduction?)

ANSWER

Take any triangle ABC. Rotate a copy of it through 180° about the midpoint of AC. The two triangles together form the quadrilateral ABCD where D is the image of B; its diagonals AC and BD bisect each other, so ABCD is a parallelogram (Theorem 4). The parallelogram is made of two triangles congruent to ΔABC.

Parallelograms tile the plane (Fig. 12.1: a grid of two sets of parallel lines). Cut each parallelogram of that tiling along a diagonal into two congruent triangles, each congruent to ΔABC.



Tiling with ΔABC (green) and its half-turn image (orange)

Therefore, copies of any triangle tile the plane.

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QUESTION 2

Mark the midpoint of the line drawn on the paper (see Fig. 12.33), given that the horizontal lines are equally spaced. Justify your answer.

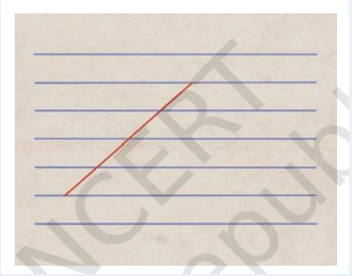
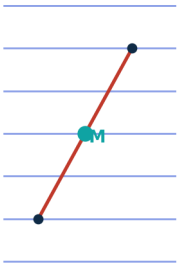


Fig. 12.33

ANSWER

The segment AB has its endpoints on two ruled lines which are 4 spaces apart (A on the lower line and B on the upper line, with 3 ruled lines between them). The **midpoint is where AB crosses the middle one of these three ruled lines**, the line that is 2 spaces from each end.



Midpoint M marked (redrawn)

Justification. Let the ruled lines through A and B be l_0 and l_4 and the middle one l_2. Draw AC perpendicular to the ruled lines, with C on l_4. The equally spaced lines cut AC into equal parts, so l_2 meets AC at its midpoint X (AX = 2 spaces, XC = 2 spaces).

In ΔACB, X is the midpoint of AC and XM ∥ CB (both on parallel ruled lines). By the Converse of the Midpoint Theorem (Theorem 7), the line through X parallel to CB bisects AB. So M, where l_2 meets AB, is its midpoint.

Therefore, the midpoint of AB is the point where it crosses the middle ruled line between its endpoints.

QUESTION 3

You know that the sum of angles of a quadrilateral is 360°, even for a non-convex quadrilateral. (Recall the proof.) Now consider a self-intersecting quadrilateral ABCD, where AB and CD intersect at point E. Show that ∠A + ∠B + ∠C + ∠D < 360°. Can you construct ABCD such that ∠A + ∠B + ∠C + ∠D = 2°?

ANSWER

Since E lies on AB and on CD, the angles of ABCD at its vertices are the angles of two triangles: ∠A = ∠DAE, ∠D = ∠ADE (in ΔAED) and ∠B = ∠CBE, ∠C = ∠BCE (in ΔBEC).

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In ΔAED: $\angle A + \angle D + \angle AED = 180°$
In ΔBEC: $\angle B + \angle C + \angle BEC = 180°$

Adding, $\angle A + \angle B + \angle C + \angle D = 360° - (\angle AED + \angle BEC)$. The angles AED and BEC are vertically opposite and so equal, say θ, with $0° < \theta < 180°$. Hence

$$\angle A + \angle B + \angle C + \angle D = 360° - 2\theta < 360°$$

b **Can the sum be 2°?** YES

We need $360° - 2\theta = 2°$, i.e. $\theta = 179°$. Draw two lines AB and CD crossing at E with $\angle AED = 179°$ (so $\angle AEC = 1°$), and take $EA = EB = EC = ED = 1$ cm. Then ΔAED and ΔBEC are isosceles with apex angle 179°, so each base angle is 0.5°, and $\angle A + \angle B + \angle C + \angle D = 4 \times 0.5° = 2°$.

Therefore, the angle sum is less than 360°, and it can be made 2°.

QUESTION 4

In a parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 12.34). Show that APCQ is a parallelogram.

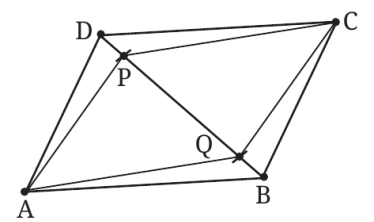

Fig. 12.34

ANSWER

Let the diagonals AC and BD meet at O. In a parallelogram the diagonals bisect each other, so OA = OC and OB = OD.

P is on OD at distance DP from D and Q is on OB at distance BQ from B. Since DP = BQ, OP = OD − DP = OB − BQ = OQ.

In quadrilateral APCQ the diagonals AC and PQ meet at O with OA = OC and OP = OQ, i.e. they bisect each other. By Theorem 4 (converse of (c)), APCQ is a parallelogram.

Therefore, APCQ is a parallelogram.

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QUESTION 5

A right-triangle shaped cutout of a paper is folded such that point A touches point B (Fig. 12.35). Show that the crease line can be used to find the midpoint of not only AB but also that of AC.

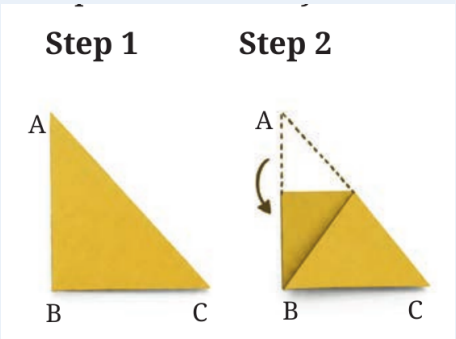

Fig. 12.35

ANSWER

The right angle of the triangle is at B (AB ⊥ BC). When A is folded onto B, the crease is the perpendicular bisector of AB. Let it meet AB at M and AC at N. Then M is the midpoint of AB (that is what the fold does).

The crease MN is perpendicular to AB, and BC is also perpendicular to AB. So MN ∥ BC.

In ΔABC, MN passes through the midpoint M of AB and is parallel to BC, so by the Converse of the Midpoint Theorem (Theorem 7) it bisects AC: N is the midpoint of AC.

Therefore, the crease meets AB at its midpoint and AC at its midpoint.

QUESTION 6

You saw how to use the Midpoint Theorem to divide a given triangle into 4 congruent triangles. Can we divide a triangle into 3 congruent triangles? This exercise shows us how to do that and more, provided we are allowed to cut and reassemble.

(i) Draw medians AP, BQ and CR of ΔABC, meeting in a common point M. Cut along each median to get 6 triangles. Show with justification how to assemble the 6 pieces into 3 congruent triangles! (**Hint:** Align ΔMPB and ΔMPC along equal sides PB and PC, matching P with itself and B with C.) Prove that you get a triangle. What are its side lengths?

*(ii) Repeat the procedure with each of the 3 assembled triangles you got in (i). Show that each of the resulting 9 triangles has sides AB/3, BC/3 and AC/3.

The procedure and the result above were unnoticed till 2014, when they were discovered by Lee Sallows, an amateur mathematician!

(iii) Naturally, we can next ask about the possibility of cutting a triangle and reassembling it into 2 congruent triangles. You were asked this question in the chapter *Measuring Space: Perimeter and Area*. Can you solve this now? (**Hint:** Use median AP in ΔABC and then a median of ΔAPB.)

ANSWER

(i) Assemble the 6 pieces into 3 congruent triangles

The medians cut ΔABC into six triangles: MPB, MPC, MQC, MQA, MRA, MRB. By the Centroid Theorem AM = 2MP, BM = 2MQ, CM = 2MR.

Pair at P. Rotate ΔMPB through 180° about P. Since P is the midpoint of BC, B goes to C, and M goes to M′ on the extension of MP with PM′ = MP. Then ΔM′PC is the image of ΔMPB.

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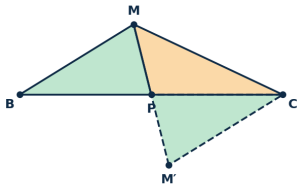

ΔMPC joined to the half-turn image ΔM′PC

It is a triangle. ∠MPC + ∠MPB = 180° (B, P, C collinear) and ∠CPM′ = ∠BPM, so ∠MPC + ∠CPM′ = 180°: M, P, M′ are collinear. Hence ΔMPC and ΔM′PC together form the triangle MM′C.

Its sides. MM′ = 2MP = AM; MC; and M′C = MB (image of MB). So the triangle MM′C has sides **AM, BM, CM**, i.e. (2/3)AP, (2/3)BQ and (2/3)CR.

Pairs at Q and R. The same construction with the pair at Q (half-turn about Q) gives a triangle with sides 2MQ = BM, MA and MC; the pair at R gives a triangle with sides 2MR = CM, MA and MB. All three triangles have sides AM, BM, CM, so they are congruent by SSS.

ii Repeat the procedure

Let T be an assembled triangle MM′C. Its medians are CP, and those to the midpoints of M′C and of MC. Taking position vectors with centroid M as origin (A + B + C = 0, so M = 0), P = (B + C)/2 = −A/2 and M′ = 2P − M = −A:

Median from C of T: C to the midpoint of MM′ = P, so its length = CP = BC/2.
Median from M: M to the midpoint of M′C = (C − A)/2, so its length = |CA|/2.
Median from M′: M′ to the midpoint of MC; (C/2) − (−A) = (C + 2A)/2 = (A − B)/2, so its length = |AB|/2.

So the medians of T are **BC/2, CA/2, AB/2**. Applying part (i) to T, each new triangle has sides equal to 2/3 of its medians: (2/3)(BC/2) = BC/3, (2/3)(CA/2) = CA/3, (2/3)(AB/2) = AB/3. Doing this for each of the 3 triangles gives 9 triangles, each with sides AB/3, BC/3, AC/3.

iii Cut into 2 congruent triangles

Let P be the midpoint of BC, so the median AP cuts ΔABC into ΔAPB and ΔAPC. Let Q be the midpoint of AB; PQ is a median of ΔAPB. Cut ΔAPB along PQ into ΔAQP and ΔBQP.

Rotate ΔBQP through 180° about Q: B goes to A, A is the image of B and P goes to P′ with Q the midpoint of PP′. Then ΔBQP maps onto ΔAQP′, and ΔAQP ∪ ΔAQP′ is the triangle APP′ (P, Q, P′ collinear).

Its sides: AP; AP′ = BP = PC; and PP′ = 2PQ = AC (Midpoint Theorem in ΔABC, PQ = AC/2). These are the sides AP, PC, AC of ΔAPC, so ΔAPP′ ≅ ΔAPC by SSS.

Therefore, ΔABC can be cut and reassembled into two congruent triangles, ΔAPC and ΔAPP′ (and into 3, and into 9, congruent triangles as in (i) and (ii)).

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QUESTION 7

A more general midpoint theorem and its converse. In a quadrilateral ABCD, suppose AB ∥ DC. Recall that such ABCD is called a trapezium. Let E be the midpoint of AD. A line drawn through E intersects side BC at F.

(i) If EF ∥ AB, then show that F is the midpoint of BC. Conclude that EF = (AB + CD)/2.

(ii) If F is the midpoint of BC, then show that EF ∥ AB.

There are at least two ways to solve (ii). Do it directly by using the midpoint M of BD and showing that EM and FM are the same lines. Or use (i) along with the same idea used in the second proof of the converse of the Midpoint Theorem. Which way do you think is simpler?

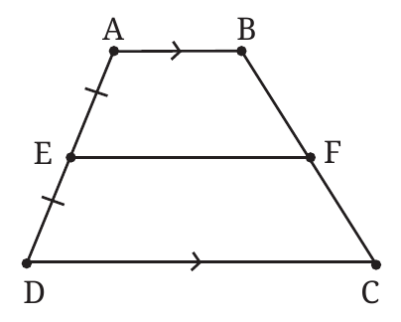

Fig. 12.36

ANSWER

(i) EF ∥ AB ⇒ F is the midpoint of BC

Since AB ∥ DC and EF ∥ AB, also EF ∥ DC. Let the diagonal AC meet EF at G. In ΔADC, E is the midpoint of AD and EG ∥ DC, so G is the midpoint of AC and EG = DC/2 (Theorem 7). In ΔACB, G is the midpoint of AC and GF ∥ AB, so F is the midpoint of BC and GF = AB/2 (Theorem 7).

$$EF = EG + GF = CD/2 + AB/2 = (AB + CD)/2$$

(ii) F is the midpoint of BC ⇒ EF ∥ AB

Direct way. Let M be the midpoint of BD. In ΔDAB, E and M are midpoints of AD and DB, so EM ∥ AB. In ΔBDC, M and F are midpoints of BD and BC, so MF ∥ DC ∥ AB. EM and MF are both lines through M parallel to AB; there is only one such line, so E, M, F lie on the same line, and EF ∥ AB.

Using (i). Draw through E the line parallel to AB; by (i) it meets BC at the midpoint of BC. But BC has only one midpoint, F, so this line passes through F, i.e. EF ∥ AB.

Both are short, but the second way (using (i) and uniqueness) is simpler because it needs no extra point.

Therefore, in a trapezium the line through the midpoint of one non-parallel side parallel to the bases bisects the other side, and EF = (AB + CD)/2; the converse also holds.

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QUESTION 8

The diagonals AC and BD of a parallelogram ABCD intersect at O. A line through O meets AB and CD at points P and Q respectively. Show that O is the midpoint of PQ. (Multiple proofs are possible. Which is the simplest?)

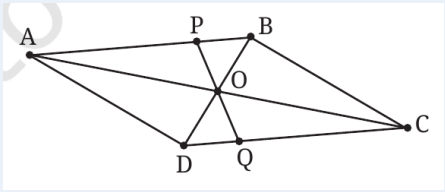

Fig. 12.37

ANSWER

In ΔOBP and ΔODQ:

OB = OD (diagonals of a parallelogram bisect each other)
∠BOP = ∠DOQ (vertically opposite angles)
∠OBP = ∠ODQ (alternate angles, AB ∥ DC, transversal BD)

So ΔOBP ≅ ΔODQ by ASA, hence OP = OQ. Since P, O, Q are collinear with O between P and Q, O is the midpoint of PQ.

Simplest proof: the half-turn about O maps the parallelogram to itself (A ↔ C, B ↔ D), so it maps line AB to line CD and the line PQ to itself; hence P goes to Q.

Therefore, O is the midpoint of PQ.

QUESTION 9 *

ABCD is a trapezium with parallel sides AD = 3 cm and BC = 5 cm. E and F are the midpoints of the non-parallel sides. Find the ratio of the areas of the 4-gons AEFD and EBCF.

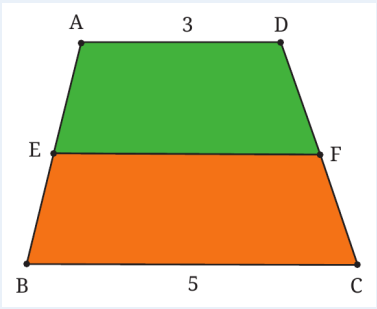

Fig. 12.38

ANSWER

E and F are the midpoints of AB and DC, so by Q7(i) EF ∥ AD ∥ BC and

EF = (AD + BC)/2 = (3 + 5)/2 = 4 cm

Let h be the distance between AD and BC. Since EF is a midline, it is at distance h/2 from both AD and BC.

Area of AEFD = ½ × (AD + EF) × h/2 = ½ × (3 + 4) × h/2 = 7h/4
Area of EBCF = ½ × (EF + BC) × h/2 = ½ × (4 + 5) × h/2 = 9h/4

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Therefore, area(AEFD) : area(EBCF) = 7 : 9.

QUESTION 10

Consider 4 points A, B, C, D in the plane with no three collinear. Answer the following questions. Some answers may require you to consider different cases, depending on how the points are positioned in the plane.

(i) How many different quadrilaterals do they form if the quadrilateral is allowed to be self-intersecting or non-convex?

(ii) How many of these quadrilaterals are self-intersecting? How many are convex?

ANSWER

i Number of quadrilaterals

A quadrilateral is fixed by the cyclic order in which the four points are joined, and ABCD, BCDA, DCBA … name the same one. Fix A: the order of B, C, D can be arranged in 3! = 6 ways, and a cycle and its reverse are the same quadrilateral, giving 6 ÷ 2 = 3. They are **ABCD, ABDC and ACBD** (for example ABCD has sides AB, BC, CD, DA).

ii Self-intersecting and convex

Case 1: the four points are the vertices of a convex quadrilateral (none lies inside the triangle of the others). Joining them in the order round the hull gives **1 convex** quadrilateral. The other two orders use the two diagonals of the hull as sides, which cross, so **2 are self-intersecting** (0 non-convex).

Case 2: one point lies inside the triangle of the other three. Whatever the order, the inside point is a dent and no sides cross: all **3 are non-convex**, **0 self-intersecting** and **0 convex**.

Therefore, there are 3 quadrilaterals: either 1 convex and 2 self-intersecting, or 3 non-convex.

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QUESTION 11

Suppose P is a point on side AB of ΔABC and the line through P parallel to BC meets AC in point Q. For parts (i) to (iii), assume AP = 1, AQ = √5, and see Fig. 12.39.

(i) If PB = 2 find QC. (**Hint:** See the dotted line)

*(ii) If PB = 1/3 find QC. (**Hint:** See the dotted lines)

*(iii) If PB = 2/3 find QC. (**Hint:** Divide both PB and AP suitably.)

*(iv) Show that if AP/PB is a rational number, then AP/PB = AQ/QC. In Grade 10, you will prove this equality without assuming AP/PB is rational. That will require a new idea.

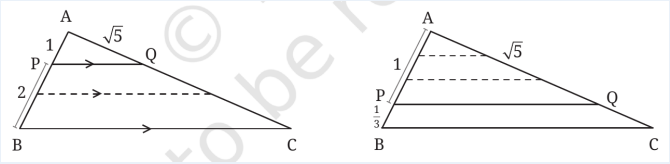

Fig. 12.39

ANSWER

Key fact (equal intercepts). Suppose points $X_0 = A, X_1, \ldots, X_n$ divide AB into equal parts and the lines through them parallel to BC meet AC at $Y_0 = A, Y_1, \ldots, Y_n$. Then $Y_0, Y_1, \ldots, Y_n$ divide AC into equal parts. *Proof:* X_i is the midpoint of $X_{i-1}X_{i+1}$ and $X_{i-1}Y_{i-1} \parallel X_iY_i \parallel X_{i+1}Y_{i+1}$. By Question 7(i) (trapezium; for i = 1 by Theorem 7 in the triangle), Y_i is the midpoint of $Y_{i-1}Y_{i+1}$. So all the parts of AC are equal.

(i) PB = 2

AB = AP + PB = 3, cut into 3 equal parts of length 1 (P is the first division point). The parallels to BC give 3 equal parts of AC, and AQ = √5 is the first. So each part = √5 and

QC = 2 parts = 2√5

(ii) PB = 1/3

Divide AP = 1 into 3 parts of 1/3. With PB = 1/3, AB is cut into 4 equal parts of 1/3 and P is the third division point. The parallels cut AC into 4 equal parts and AQ is 3 of them: 3 parts = √5, so one part = √5/3 and

QC = 1 part = √5/3

(iii) PB = 2/3

Divide AP = 1 = 3/3 into 3 parts of 1/3 and PB = 2/3 into 2 parts of 1/3. AB is cut into 5 equal parts, P being the third division point. AC is cut into 5 equal parts and AQ is 3 of them: one part = √5/3 and

QC = 2 parts = 2√5/3

(iv) AP/PB rational ⇒ AP/PB = AQ/QC

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Let AP/PB = m/n with positive integers m, n. Divide PB into n equal parts of length u. Then AP = (m/n)PB = mu, so AP can be divided into m parts of length u. Thus AB is divided into m + n equal parts and P is the m-th division point from A.

Draw through every division point the line parallel to BC. By the key fact AC is cut into m + n equal parts, say of length v. The line through P parallel to BC is PQ (it is unique), so Q is the m-th division point from A: AQ = mv and QC = nv. Hence

$$AQ/QC = m/n = AP/PB$$

Therefore, QC = 2√5, √5/3 and 2√5/3 in (i), (ii), (iii), and AP/PB = AQ/QC whenever AP/PB is rational.

QUESTION 12

(i) Suppose ABCD is a parallelogram and M, N are midpoints of AB and CD respectively. Show that segments DM and BN trisect segment AC.

(ii) Use part (i) to find a procedure to trisect any given segment PQ. Find two other ways to trisect PQ, one using Exercise 11 above and a third using the Centroid Theorem.

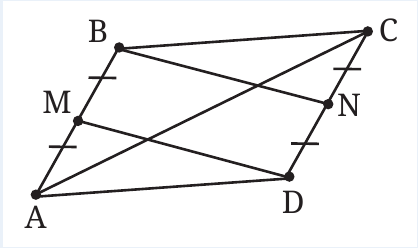

Fig. 12.40

ANSWER

i DM and BN trisect AC

MB = AB/2 = DC/2 = DN and MB ∥ DN, so MBND is a parallelogram and DM ∥ BN. Let DM and BN meet AC at X and Y.

In ΔABY, M is the midpoint of AB and MX ∥ BY, so X is the midpoint of AY (Theorem 7): AX = XY. In ΔCDX, N is the midpoint of CD and NY ∥ DX, so Y is the midpoint of CX: XY = YC.

$$AX = XY = YC$$

ii Trisecting a segment PQ

Way 1 (part (i)). Take any point B not on PQ; draw through P a line parallel to QB and through Q a line parallel to PB, meeting at D, so PBQD is a parallelogram with diagonal PQ (A = P, C = Q). Find the midpoints M of PB and N of QD. Then DM and BN meet PQ at the two points that trisect it.

Way 2 (Exercise 11). Draw a ray from P (not along PQ) and mark on it three equal lengths $PX_1 = X_1X_2 = X_2X_3$ with a compass. Join X_3 to Q and draw through X_1 and X_2 lines parallel to X_3Q. By the equal-intercept fact of Exercise 11 they cut PQ into three equal parts.

Way 3 (Centroid Theorem). Take PQ as a median of a triangle PBC: draw a line through Q and mark B, C on it with QB = QC (using a compass). Draw the median from B to the midpoint of PC; it meets PQ at the centroid G with PG : GQ = 2 : 1, so GQ = PQ/3. The midpoint of PG is the other trisection point.

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Therefore, DM and BN trisect AC, and PQ can be trisected in three ways.

QUESTION 13

Is the Midpoint Theorem for Quadrilaterals (Theorem 9) true when the quadrilateral is non-convex? How about when it is self-intersecting? Experiment and check. The pictures may look strange, but the theorem appears to be true.

(i) Can you explain why this is so? See the geometric reasoning in the text.

(ii) There is one exception: In a very special case the Varignon parallelogram becomes a single segment. When will this happen?

*(iii) Does your reasoning apply even when ABCD is non-planar?

ANSWER

i Why the theorem holds

TRUE

The proof in the text only uses the Midpoint Theorem in the triangles ABC, ADC, BCD and BAD: PQ ∥ AC and SR ∥ AC, so PQ ∥ SR (and PQ = SR = AC/2); also QR ∥ BD ∥ PS (QR = PS = BD/2). The Midpoint Theorem holds for every triangle, wherever the fourth point lies. No triangle used is degenerate (no three of A, B, C, D are collinear), and the proof never uses that ABCD is convex or non-self-intersecting. So PQRS is a parallelogram in the non-convex and in the self-intersecting case too.

ii The exception

PQ ∥ AC and QR ∥ BD. If the diagonals AC and BD are **parallel**, then PQ and QR lie along the same direction, so P, Q, R (and S) are collinear and PQRS collapses into a segment. This can only happen for a self-intersecting quadrilateral: in a convex quadrilateral the diagonals intersect, and in a non-convex one the line through the dent vertex cuts the opposite diagonal. For example A(0, 0), C(2, 0), B(1, 1), D(3, 1): AC ∥ BD and the midpoints (0.5, 0.5), (1.5, 0.5), (2.5, 0.5), (1.5, 0.5) lie on one line.

iii Non-planar ABCD

YES

The Midpoint Theorem is applied to the triangles ABC and ADC, which are planar even if ABCD is not. So PQ ∥ AC ∥ SR, with PQ = SR = AC/2. Two parallel equal segments PQ and SR lie in one plane and form a parallelogram (Theorem 5), so P, Q, R, S are coplanar and PQRS is a parallelogram.

Therefore, Varignon's theorem holds for non-convex, self-intersecting and even non-planar quadrilaterals, with the segment as the only exceptional case.

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QUESTION 14

Let P, Q, R, S be four points on sides AB, BC, CD and DA respectively of a quadrilateral ABCD. Suppose PQRS is a parallelogram. Must P, Q, R and S be midpoints of the respective sides? This can be considered a possible converse question to Theorem 9.

ANSWER

No. We give a counterexample. Take the square A(0, 0), B(4, 0), C(4, 4), D(0, 4) and choose

$$P = (1, 0) \text{ on } AB\ (AP = 1),\ Q = (4, 1) \text{ on } BC\ (BQ = 1),$$
$$R = (3, 4) \text{ on } CD\ (CR = 1),\ S = (0, 3) \text{ on } DA\ (DS = 1).$$

Then PQ = (3, 1) = SR (equal and parallel) and QR = (−1, 3) = PS, so PQRS is a parallelogram (in fact PQ ⊥ QR and PQ = QR = √10, so it is a square). But P, Q, R, S are not the midpoints of the sides (AP = 1 while AB = 4).

In general, in any parallelogram ABCD, taking AP = CR and BQ = DS makes PQRS a parallelogram, because the half-turn about the centre of ABCD maps P to R and Q to S, so PR and QS bisect each other.

Therefore, no: P, Q, R, S need not be midpoints.

QUESTION 15

(i) Complete the following proof of the Midpoint Theorem. Extend segment PQ beyond Q until point S. By how much should we extend PQ? It would be good to be able to prove that ΔAPQ ≅ ΔCSQ. (Why?) Use this as a guide to specify the location of S and then complete this proof.

(ii) Give a similar proof of the converse of the Midpoint Theorem (Theorem 7). Start by extending PQ up to a suitable point S.

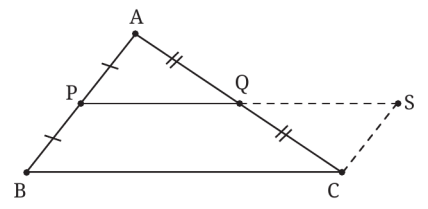

Fig. 12.41

ANSWER

i Proof of the Midpoint Theorem

P, Q are the midpoints of AB, AC. **Extend PQ beyond Q to S so that QS = PQ** (extend by the length PQ). Join CS.

In ΔAPQ and ΔCSQ: AQ = CQ (Q is the midpoint), ∠AQP = ∠CQS (vertically opposite angles), PQ = SQ (by construction). So ΔAPQ ≅ ΔCSQ by SAS. (This is why this is the right place for S.)

Hence CS = AP = PB and ∠APQ = ∠CSQ. These are alternate angles for AB and CS with transversal PS, so CS ∥ AB, i.e. CS ∥ PB. CS and PB are equal and parallel, so by Theorem 5 PBCS is a parallelogram. Then PS ∥ BC and PS = BC. Since PQ = PS/2:

$$PQ \parallel BC \text{ and } PQ = BC/2$$

ii Proof of the converse (Theorem 7)

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P is the midpoint of AB; PQ ∥ BC with Q on AC. Extend PQ beyond Q up to the point S where the line through C parallel to BA meets it (S is on the line PQ and CS ∥ AB).

PBCS has PS ∥ BC and CS ∥ BP, so it is a parallelogram and CS = BP = AP. In ΔAPQ and ΔCSQ: ∠APQ = ∠CSQ (alternate angles, AB ∥ CS), ∠AQP = ∠CQS (vertically opposite) and AP = CS. So ΔAPQ ≅ ΔCSQ by AAS.

Therefore AQ = CQ, i.e. Q is the midpoint of AC, and PQ = QS = PS/2 = BC/2.

Therefore, QS = PQ is the required extension, and the converse is proved similarly.

QUESTION 16 *

Review all the properties of a rhombus/rectangle/square that you proved in Grade 8. Formulate a converse of each. Decide if the converse is true. There are many possibilities here!

ANSWER

R Rhombus

P: All four sides of a rhombus are equal. Converse: a quadrilateral with equal sides is a rhombus. TRUE (opposite sides equal ⇒ parallelogram, adjacent sides equal).
P: The diagonals of a rhombus are perpendicular. Converse: a quadrilateral with perpendicular diagonals is a rhombus. FALSE: a kite has perpendicular diagonals. It becomes TRUE for a parallelogram, or if the diagonals also bisect each other (Ex. 12.2 Q3(ii)).
P: The diagonals of a rhombus bisect its angles. Converse: if the diagonals of a quadrilateral bisect its angles, it is a rhombus. TRUE (Ex. 12.2 Q3(i)).

C Rectangle

P: All angles of a rectangle are 90°. Converse: a quadrilateral with all angles equal is a rectangle. TRUE (equal angles are 90° each, so opposite angles are equal ⇒ parallelogram with a right angle).
P: The diagonals of a rectangle are equal. Converse: a quadrilateral with equal diagonals is a rectangle. FALSE: an isosceles trapezium. TRUE for a parallelogram (Ex. 12.2 Q1(iii)) or if the equal diagonals bisect each other.
P: A rectangle's diagonals bisect each other. Converse: if the diagonals bisect each other, it is a rectangle. FALSE: a parallelogram that is not a rectangle. Adding "and are equal" makes it TRUE.

S Square

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P: The diagonals of a square are equal, perpendicular and bisect each other. Converse: if the diagonals of a quadrilateral are equal and perpendicular, it is a square. FALSE (example: A(0, 0), C(4, 0), B(1, 2), D(1, −2); AC = BD = 4, AC ⊥ BD, but AB = √5 ≠ BC = √13). TRUE if the diagonals also bisect each other.
P: All sides of a square are equal and all angles 90°. Converse: a quadrilateral with equal sides and equal angles is a square. TRUE.
P: A square is a rectangle and a rhombus. Converse: a quadrilateral that is both a rectangle and a rhombus is a square. TRUE (right angles + equal sides).

Therefore, each converse is either true (and proved above), or false with the counterexample and the extra condition that repairs it.

QUESTION 17 *

Show that the sum of angles of a non-planar quadrilateral is always less than 360°. Can you find a non-planar quadrilateral ABCD for which ∠A + ∠B + ∠C + ∠D = 2°?

(Hint: Think of a diagonal, say AC, as a hinge around which triangles ABC and ADC can rotate.) What happens to each angle of ABCD as you do this rotation?

ANSWER

Use AC as a hinge. Triangles ABC and ADC keep their shapes, so ∠B, ∠D, ∠BAC, ∠CAD, ∠BCA, ∠ACD never change. Let φ be the angle between the two planes (φ = 180° when the triangles lie in the same plane on opposite sides of AC, so ABCD is flat).

Flat position (φ = 180°). Let B_0 be the position of B. The flat quadrilateral AB_0CD has angle sum 360°.

Rotating. B moves on a circle around the axis AC. If F and H are the feet of B and D on AC and r = BF, s = DH, then $BD^2 = FH^2 + r^2 + s^2 - 2rs\cdot\cos\varphi$. So BD **decreases** as φ decreases from 180°.

In ΔABD, AB and AD are fixed, so a smaller BD means a smaller angle: ∠BAD decreases (and likewise ∠BCD in ΔCBD). The other two angles ∠B, ∠D do not change. So the sum ∠A + ∠B + ∠C + ∠D, which is 360° when flat, becomes strictly smaller as soon as ABCD is non-planar.

Therefore, the sum of the angles of a non-planar quadrilateral is always less than 360°.

b A non-planar ABCD with angle sum 2°

YES

Take an isosceles triangle ABC with AB = CB and ∠B = 0.5° (so ∠A = ∠C = 89.75°) and let ADC be a congruent copy (∠D = 0.5°). Rotate ADC about AC. When φ = 0° the triangles coincide, the angle at A and C is 0° and the sum is 0.5° + 0.5° = 1°. When φ = 180° (flat), ∠BAD + ∠BCD = 359°.

As φ increases from 0° to 180°, ∠BAD + ∠BCD increases continuously from 0° to 359°, so for some φ strictly between, it equals 1°, and then ∠A + ∠B + ∠C + ∠D = 1° + 1° = 2°. This ABCD is non-planar because φ ≠ 0° and φ ≠ 180°.

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Therefore, such a non-planar quadrilateral exists.

QUESTION 18 *

Let us see a third method to tile the plane using a 4-gon. Focus on only two *coloured copies* of SOME sharing a vertex. What do you see? It appears that each 4-gon is just a shifted copy of the other. Let us see exactly how. Draw two copies of SOME as shown in Fig. 12.42 so that the diagonals EO and E′O′ are collinear with O = E′. Now place a cutout of one copy on top of SOME with diagonal EO drawn on it. Slide this cutout so that segment EO moves along EO′ until EO matches E′O′. Verify that your cutout exactly matches S′O′M′E′.

(i) Prove the exact match of SOME with S′O′M′E′ using four parallelograms.

(ii) Follow the described procedure along each diagonal of the starting coloured 4-gon in both directions to get 4 new 4-gons. Using these 4 copies, repeat the procedure 4 more times to place 4 new copies, resulting in a 3 by 3 grid of 9 copies exactly like the 9 coloured copies in Fig. 12.42. These 9 copies meet each other only at vertices and once again the blank spaces among them also form 4-gons congruent to SOME!

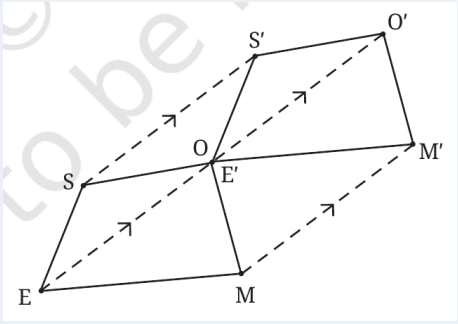

Fig. 12.42

ANSWER

(i) Exact match with four parallelograms

The 4-gon S′O′M′E′ is obtained by sliding SOME along EO′ by the vector EE′ = EO: the segments SS′, EE′ (= EO), OO′ (= E′O′) and MM′ are parallel (all along the direction EO) and equal in length (each equals EO, since EO = E′O′ and E′ = O).

Consider the four quadrilaterals ESS′E′, SOO′S′, OMM′O′ and MEE′M′. In each, two opposite sides (SS′ and EE′; SS′ and OO′; OO′ and MM′; MM′ and EE′) are equal and parallel, so by Theorem 5 each is a **parallelogram**.

Therefore their other opposite sides are equal and parallel:

S′E′ = SE (ESS′E′), S′O′ = SO (SOO′S′),
O′M′ = OM (OMM′O′), M′E′ = ME (MEE′M′),
and E′O′ = EO.

So the sides of S′O′M′E′ equal those of SOME and the diagonal E′O′ = EO. Hence ΔS′O′E′ ≅ ΔSOE and ΔM′O′E′ ≅ ΔMOE (SSS), so all the angles match, and S′O′M′E′ ≅ SOME. Because it is obtained by sliding, it is a shifted copy.

(ii) The 3 × 3 grid

Sliding along EO and OE gives the copies SOME ± EO; sliding along SM and MS gives SOME ± SM: 4 new copies. Slide each of these along the other diagonal: the 4 new copies SOME ± EO ± SM. Together with SOME these are the 9 copies SOME + i·EO + j·SM, i, j ∈ {−1, 0, 1}, i.e. a 3 × 3 grid.

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The translation vectors are the diagonals EO and SM (d_1 and d_2 of Method 1), so these 9 copies are the green copies of the tiling of Fig. 12.28 (the set $Q + i \cdot d_1 + j \cdot d_2$). Two of them touch only at vertices, and the gaps between them are the half-turn images H(SOME) shifted by the same vectors; each gap is therefore a 4-gon congruent to SOME.

Therefore, S′O′M′E′ is a shifted copy of SOME, and the sliding procedure produces the same tiling as Methods 1 and 2.

QUESTION 19 *

Is there a 4-gon with given side lengths?

(i) Recall the following fact about triangles and check it by construction. For given positive numbers a, b, c, is there a triangle whose sides have these lengths? The answer is Yes exactly when the sum of any two numbers is greater than the third. If we arrange the numbers in increasing order (suppose $a \le b \le c$), then this amounts to requiring $a + b > c$. (**Hint:** Start by drawing a segment of length c.)

(ii) Suppose a 4-gon has 2, 5, 11 as three side lengths. Can the length of the fourth side be 100? Can it be 10? Can it be 1? What are the possible lengths of the fourth side?

(iii) For given positive numbers a, b, c, d, how will you decide if there is a 4-gon whose sides have these lengths?

ANSWER

i Triangle with sides $a \le b \le c$

Draw a segment XY of length c. With centre X draw a circle of radius b and with centre Y a circle of radius a. A point Z on both circles gives a triangle XYZ with sides c, b, a. Two circles with radii b, a and centres c apart meet in two points exactly when $b - a < c < a + b$. Since $b \le c$, the condition $b - a < c$ always holds, so the construction works exactly when **$a + b > c$**. The other inequalities $a + c > b$ and $b + c > a$ then hold automatically.

ii Sides 2, 5, 11 and x

Draw the diagonal of the 4-gon; it splits it into two triangles. A 4-gon exists exactly when each side is shorter than the sum of the other three (by the triangle inequality, with the diagonal as the third side: $x < 2 + 5 + 11 = 18$ and $11 < 2 + 5 + x$, i.e. $x > 4$).

$x = 100$: $100 > 18$, so NO
$x = 10$: $4 < 10 < 18$, so YES
$x = 1$: $1 < 4$, so NO ($2 + 5 + 1 = 8 < 11$)

The possible lengths of the fourth side are **$4 < x < 18$**.

iii Sides a, b, c, d

Arrange them in increasing order $a \le b \le c \le d$. A 4-gon with these sides exists if and only if the largest side is less than the sum of the other three:

$$d < a + b + c$$

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(If d ≥ a + b + c, the three shorter sides cannot reach across; if d < a + b + c, choose a diagonal of a suitable length so that both triangles exist.)

Therefore, a 4-gon with sides a, b, c, d exists exactly when each side is less than the sum of the other three, i.e. the longest side < sum of the other three.

QUESTION 20 *

Counting diagonals of a polygon.

(i) How should we define a diagonal of an *n*-gon? How many diagonals does an *n*-gon have? Make a table for small values of *n*. A 3-gon has no diagonals. A 4-gon has 2. How many diagonals does a 5-gon have? A 6-gon? Try to find a pattern and guess the answers for *n* = 7 and *n* = 8. Check your guesses by systematic counting.

(ii) Can you guess a formula for the number of diagonals? How many diagonals get added when we increase the number of sides by 1? Can you now justify why the formula you guessed is true for all *n*?

ANSWER

i Definition and table

A **diagonal** of an n-gon is a segment joining two vertices that are not adjacent (that is not a side). Each vertex is joined to n − 3 others by diagonals (not itself, not its two neighbours).

n = 3: 0 n = 4: 2 n = 5: 5 n = 6: 9
n = 7: 14 n = 8: 20

The increase from n to n + 1 is 2, 3, 4, 5, 6, … The guesses are n = 7: 9 + 5 = 14 and n = 8: 14 + 6 = 20, which agree with systematic counting (n = 7: 7 × 4/2 = 14; n = 8: 8 × 5/2 = 20).

ii Formula

Number of diagonals = n(n − 3)/2

Justification 1. From each of the n vertices there are n − 3 diagonals, which gives n(n − 3). Each diagonal has two ends and has been counted twice, so divide by 2.

Adding one side. To go from an n-gon to an (n + 1)-gon, insert a new vertex Z between two adjacent vertices X and Y. The old side XY becomes a diagonal (+1), and Z is joined by diagonals to all vertices except itself and its two neighbours X, Y, i.e. to n − 2 vertices (+(n − 2)). So n − 1 diagonals are added. The formula agrees: (n + 1)(n − 2)/2 − n(n − 3)/2 = (2n − 2)/2 = n − 1, and it starts correctly (n = 3 gives 0). Hence it is true for all n.

Therefore, an n-gon has n(n − 3)/2 diagonals.

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QUESTION 21 *

Sum of angles of a polygon. What is the sum of angles of a (planar non-self-intersecting) *n*-gon? We know that the answer is 180° for *n* = 3 and 360° for *n* = 4. Find the next few values. Then find a formula in terms of *n* and prove it.

ANSWER

Values: a triangle can be cut by a diagonal into pieces; every polygon with n ≥ 4 sides has a diagonal lying inside it that cuts it into two smaller polygons. Using 180° for n = 3 and 360° for n = 4:

$$n = 3: 180° \; n = 4: 360° \; n = 5: 540° \; n = 6: 720° \; n = 7: 900°$$

$$\text{Sum of the angles} = (n - 2) \times 180°$$

Proof (by induction on n). The formula holds for n = 3. Let n ≥ 4 and assume it for all polygons with fewer sides. Cut the n-gon by an inside diagonal into a polygon with m sides and one with k sides, where m + k = n + 2 (the diagonal is a side of both), m, k < n. The angles at the two ends of the diagonal are split between the two pieces, so the angles of the n-gon together make up exactly the angles of the two pieces:

$$(m - 2) \times 180° + (k - 2) \times 180° = (m + k - 4) \times 180° = (n - 2) \times 180°$$

Therefore, the angle sum of an n-gon is (n − 2) × 180°.

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QUESTION 22 *

Multiple converses to a theorem. Let us see how multiple statements can be considered converses to the Midpoint Theorem and how Theorem 7 is one of them.

To formulate a converse we should express the original statement in "If ... then ..." form. For a complex statement, there may be multiple ways to do that. The Midpoint Theorem starts with ΔABC and points P and Q on sides AB and AC respectively. The theorem has two assumptions and two conclusions, which we have named for further discussion.

Assumptions: (P MID) P is the midpoint of AB, and (Q MID) Q is the midpoint of AC.

Conclusions: (PRLL) PQ ∥ BC, and (HALF) PQ = BC/2.

Let us use these four named conditions to discuss various possible statements.

(i) A natural converse of the Midpoint Theorem would be: "If (PRLL) and (HALF) are true, then (P MID) and (Q MID) are true." Write out this statement fully. Experiment and see that it seems to be true! Prove the statement. (**Hint:** Try to run a proof of the Midpoint Theorem backwards.)

(ii) To see Theorem 7 as a converse of the Midpoint Theorem, we first write the Midpoint Theorem as follows: "Suppose P is the midpoint of side AB of ΔABC and Q is a point on side AC. If Q is the midpoint of AC then PQ ∥ BC." (Also PQ = BC/2, but let us set that aside for now.) Write the converse of the second sentence in quotes by keeping the first sentence the same. Verify that Theorem 7 is what you get! (And PQ = BC/2 also follows.) Now write Theorem 7 in terms of the four named conditions.

(iii) The discussion so far suggests that it is reasonable to take two of the four listed statements as assumptions and ask if the other two are true. Verify that only one such combination remains to be examined. "If (P MID) and (HALF) are true, then can we conclude (Q MID) and/or (PRLL)?" Write this out in words. This is a precise version of question (1) stated before Theorem 7. Can you answer it?

In the text, why did we focus only on Theorem 7 among the multiple possibilities? Because Theorem 7 turns out to be particularly useful. It is still important to pursue various avenues while exploring a subject: often it is only after examining multiple possibilities that we can judge what is most valuable.

ANSWER

i (PRLL) and (HALF) ⇒ (P MID) and (Q MID) — TRUE

Statement: "In ΔABC, let P be a point on AB and Q a point on AC. If PQ ∥ BC and PQ = BC/2, then P is the midpoint of AB and Q is the midpoint of AC."

Proof (running the proof backwards). Extend PQ beyond Q to S with QS = PQ, and join CS. Then PS = 2PQ = BC and PS ∥ BC, so PBCS has a pair of opposite sides equal and parallel: it is a parallelogram, so CS = BP and CS ∥ BP (i.e. CS ∥ AB). In ΔAPQ and ΔCSQ: ∠APQ = ∠CSQ (alternate angles), ∠AQP = ∠CQS (vertically opposite), PQ = SQ. So ΔAPQ ≅ ΔCSQ (AAS). Hence AQ = CQ and AP = CS = BP: Q is the midpoint of AC and P is the midpoint of AB.

ii Theorem 7 as a converse

The second sentence is "If Q is the midpoint of AC then PQ ∥ BC". Its converse, with the first sentence kept, is: "Suppose P is the midpoint of side AB of ΔABC and Q is a point on side AC. If PQ ∥ BC then Q is the midpoint of AC." This is exactly Theorem 7 (the line through the midpoint of one side parallel to another side bisects the third side), and PQ = BC/2 follows.

In terms of the named conditions: "If (P MID) and (PRLL) are true, then (Q MID) (and (HALF)) are true."

iii The remaining combination

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Choosing two of the four conditions as assumptions gives 6 possibilities: {P MID, Q MID} (the Midpoint Theorem), {PRLL, HALF} (part (i)), {P MID, PRLL} and {Q MID, PRLL} (Theorem 7, symmetric), {Q MID, HALF} and {P MID, HALF}. By symmetry between P and Q only {P MID, HALF} (equivalently {Q MID, HALF}) remains.

In words: "In ΔABC, if P is the midpoint of AB and Q is a point on AC with PQ = BC/2, must Q be the midpoint of AC, and/or must PQ be parallel to BC?"

Answer: No. Take A = (0, 0), B = (5, 4), C = (6, 0) and P = (2.5, 2), the midpoint of AB. Let Q = (2, 0) on AC. Then PQ = √(0.25 + 4) = √17/2 = BC/2, so (P MID) and (HALF) hold. But Q is not the midpoint (3, 0) of AC, and PQ is not parallel to BC (PQ has direction (−0.5, −2), BC has direction (1, −4)).

Hence from (P MID) and (HALF) alone we cannot conclude (Q MID) or (PRLL).

Therefore, (i) is true, Theorem 7 is the converse in (ii), and the case of (P MID) and (HALF) in (iii) does not give a converse: neither conclusion need hold.

QUESTION 23 *

Validity of tiling methods. Show using reasoning that each of the three tiling methods we saw produces a tiling of the plane using the given 4-gon. You have to prove that when we place new copies using any of the three procedures, the entire plane is covered with no gaps and no overlaps. First, think carefully about what it means to prove this. Then find a proof for each procedure.

ANSWER

What must be proved. (a) Every point of the plane lies in some copy (no gaps). (b) No two copies share interior points (no overlaps).

1 Method 1 (half-turns)

Let the 4-gon be $V_0V_1V_2V_3$ and H the half-turn about the midpoint of V_0V_1, so $H(x) = V_0 + V_1 - x$. Put $d_1 = V_2 - V_0$ and $d_2 = V_3 - V_1$.

Neighbours. The copy across edge V_aV_b of a copy Q is the half-turn image $(V_a + V_b) - Q$. For Q = SOME this equals H(Q) + t with t = 0 (edge V_0V_1), d_1 (edge V_1V_2), $d_1 + d_2$ (edge V_2V_3) and d_2 (edge V_3V_0). For H(Q), the neighbours across its edges are Q − t with the same t. Hence the set Ω of copies Q + t and H(Q) + t, with $t = i \cdot d_1 + j \cdot d_2$ (i, j integers), has the property: *the copy across every edge of every member of Ω is again in Ω, and it fits exactly along that edge*.

Angles at a vertex. Because $V_2 = V_0 + d_1$, $V_3 = V_1 + d_2$, $H(V_1) = V_0$ and $H(V_0) = V_1$, the vertices of the pattern are of two kinds: $V_0 + t$ and $V_1 + t$. At the point V_0, the copies of Ω that have a vertex there are: Q (angle $\angle V_0$), $Q - d_1$ (its corner V_2 lands on V_0: angle $\angle V_2$), H(Q) (its corner $H(V_1) = V_0$: angle $\angle V_1$) and $H(Q) + d_2$ (its corner $H(V_3) = V_0 - d_2$ lands on V_0 after the shift by d_2: angle $\angle V_3$). So the four angles 1, 2, 3, 4 occur once each and add up to 360°. The same holds at a vertex of the other kind.

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No gaps. Let U be the union of all the copies in Ω. U is closed (the copies are closed and only finitely many meet any bounded region). U is also open: around any point of a copy, copies of Ω fill a full neighbourhood: inside the copy trivially, across an edge by the neighbour property, and at a vertex by the angle sum 360°. The plane is connected, so the non-empty set U which is both open and closed is the whole plane.

No overlaps. The pattern repeats under translations by $i \cdot d_1 + j \cdot d_2$. One cell of this grid has area $|d_1 \times d_2| = 2 \times$ area(SOME) (the cross product of the diagonals), and contains exactly two copies. They cover the cell with total area equal to the cell's area, so they cannot overlap.

2 Method 2 (Varignon grid)

The Varignon parallelogram Π of Q has sides $u = d_1/2$ and $w = d_2/2$ (PQ ∥ AC, QR ∥ BD with half lengths). The grid of Π is the grid of parallelograms with sides u, w. The shaded cells in Fig. 12.32 are the cells Π + t, $t = i \cdot d_1 + j \cdot d_2$ (every second cell in both directions). A copy of SOME placed in its own cell as the original sits in Π is Q + t. So the placed copies are exactly the copies Q + t of Method 1. They meet only at vertices; the gaps left are, by Method 1, the copies H(Q) + t, which are 4-gons congruent to SOME. So Method 2 gives the pattern Ω of Method 1, which is a tiling.

3 Method 3 (sliding along diagonals)

Sliding SOME along its diagonal EO by EO and along SM by SM produces the copies $Q + i \cdot d_1 + j \cdot d_2$ (here d_1, d_2 are the diagonal vectors), shown in Exercise 18 to be exact copies. These are the translates Q + t of Method 1, and the gaps are the H(Q) + t. So Method 3 also produces Ω.

Therefore, all three procedures produce the same pattern, which covers the plane with no gaps and no overlaps.

QUESTION 24 *

What fraction of the square is shaded?

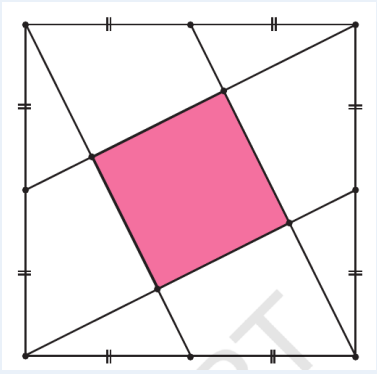

Fig. 12.43

ANSWER

From the figure the midpoints of the four sides are joined to the opposite vertices as shown (each side is divided into two equal parts). Take a square of side 2 with A(0, 0), B(2, 0), C(2, 2), D(0, 2) and the four lines

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L_1: from D(0, 2) to the midpoint (1, 0) of AB.
L_2: from the midpoint (1, 2) of DC to B(2, 0).
L_3: from C(2, 2) to the midpoint (0, 1) of AD.
L_4: from A(0, 0) to the midpoint (2, 1) of BC.

L_1 has direction (1, −2) and L_3 has direction (−2, −1); since 1 × (−2) + (−2) × (−1) = 0, $L_1 \perp L_3$. In the same way $L_1 \parallel L_2$ (both have direction (1, −2)) and $L_3 \parallel L_4$. So the four lines bound a rectangle. Its vertices are the intersections:

$L_1 \cap L_3 = (0.4, 1.2)$, $L_3 \cap L_2 = (1.2, 1.6)$,
$L_2 \cap L_4 = (1.6, 0.8)$, $L_4 \cap L_1 = (0.8, 0.4)$.

Its sides are $\sqrt{((1.2 - 0.4)^2 + (1.6 - 1.2)^2)} = \sqrt{0.8}$ and $\sqrt{((1.6 - 1.2)^2 + (0.8 - 1.6)^2)} = \sqrt{0.8}$ (equal), so it is a square of area 0.8.

Fraction shaded = 0.8 / 4 = 1/5

Therefore, 1/5 of the square is shaded.

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